

Generalized functions beyond distributions

Vieri Benci* Lorenzo Luperi Baglini†

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Abstract

Ultrafunctions are a particular class of functions defined on a Non Archimedean field $\mathbb{R}^* \supset \mathbb{R}$. They have been introduced and studied in some previous works ([1],[2],[3]). In this paper we introduce a modified notion of ultrafunction and we discuss sistematically the properties that this modification allows. In particular, we will concentrated on the definition and the properties of the operators of derivation and integration of ultrafunctions.

Keywords. Ultrafunctions, Delta function, distributions, Non Archimedean Mathematics, Non Standard Analysis.

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*Dipartimento di Matematica, Università degli Studi di Pisa, Via F. Buonarroti 1/c, 56127 Pisa, ITALY and Department of Mathematics, College of Science, King Saud University, Riyadh, 11451, SAUDI ARABIA. e-mail: benci@dma.unipi.it

†University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090 Vienna, AUSTRIA, e-mail: lorenzo.luperi.baglini@univie.ac.at, supported by grant P25311-N25 of the Austrian Science Fund FWF.

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1 Introduction

In some recent papers the notion of ultrafunction has been introduced and studied ([1], [2], [3]). Ultrafunctions are a particular class of functions defined on a Non Archimedean field $\mathbb{R}^* \supset \mathbb{R}$. We recall that a Non Archimedean field is an ordered field which contains infinite and infinitesimal numbers. In general, as we showed in our previous works, when working with ultrafunctions we associate to any continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ an ultrafunction $\tilde{f} : (\mathbb{R}^*)^N \rightarrow \mathbb{R}^*$ which extends f ; more exactly, to any vector space of functions $V(\Omega) \subseteq L^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ we associate a space of ultrafunctions $\tilde{V}(\Omega)$. The spaces of ultrafunctions are much larger than the corrspective spaces of functions, and have much more "compactness": these two properties ensure that in the spaces of ultrafunctions we can find solutions to functional equations which do not have any solutions among the real functions or the distributions.

In [3] we studied the basic properties of ultrafunctions. One property that is missing, in general, is the "locality": local changes to an ultrafunction (namely, changing the value of an ultrafunction in a neighborhood of a point) affects the ultrafunction globally (namely, they may force to change the values of the ultrafunction in all the points). This problem is related to the properties of a particular basis of the spaces of ultrafunctions, called "Delta basis" (see [2], [3]). The elements of a Delta basis are called Delta ultrafunctions and, in some precise sense, they are an analogue of the Delta distributions. More precisely, given a point $a \in \mathbb{R}^*$, the Delta ultrafunction centered in a (denoted by $\delta_a(x)$) is the unique ultrafunction such that, for every ultrafunction $u(x)$, we have

$$\int^* u(x) \delta_a(x) dx = u(a).^1$$

It would be useful for applications to have an orthonormal Delta basis, namely a Delta basis $\{\delta_a(x)\}_{a \in \Sigma}$ such that, for every $a, b \in \Sigma$, $\int^* \delta_a(x) \delta_b(x) dx = \delta_{a,b}$; unfortunately, this seems to be impossible.

The main aim of this paper is to show how to modify the constructions exposed in [3] (that will be recalled) to avoid such unwanted issues. We will show how to construct spaces of ultrafunctions that have "good local properties" and that have Delta bases $\{\delta_a(x)\}_{a \in \Sigma}$ that are "almost orthogonal" where, by saying that a Delta basis is "almost orthogonal", we mean the following: for every $a, b \in \Sigma$, if $|a - b|$ is not infinitesimal² then $\int^* \delta_a(x) \delta_b(x) dx = 0$.

¹ $\int^* : L^1(\mathbb{R})^* \rightarrow \mathbb{C}^*$ is an extension of the integral $\int : L^1(\mathbb{R}) \rightarrow \mathbb{C}$.

²We recall that an element x of a Non Archimedean superreal ordered field $\mathbb{K} \supset \mathbb{R}$ is infinitesimal if $|x| < r$ for every $r \in \mathbb{R}$.

We will also discuss a few other properties of ultrafunctions that were missing in the previous approach but that hold in this new context.

The techniques on which the notion of ultrafunction is based are related to Non Archimedean Mathematics (NAM) and to Nonstandard Analysis (NSA). In particular, the most important notion that we use is that of Λ -limit (see [1], [2], [3]). In this paper this notion will be considered known; however, for sake of completeness, we will recall its basic properties in the Appendix.

1.1 Notations

If X is a set then

- $\mathcal{P}(X)$ denotes the power set of X and $\mathcal{P}_{fin}(X)$ denotes the family of finite subsets of X ;
- $\mathfrak{F}(X, Y)$ denotes the set of all functions from X to Y and $\mathfrak{F}(\mathbb{R}^N) = \mathfrak{F}(\mathbb{R}^N, \mathbb{R})$.

Let Ω be a subset of \mathbb{R}^N : then

- $\mathcal{C}(\Omega)$ denotes the set of continuous functions defined on $\Omega \subset \mathbb{R}^N$;
- $\mathcal{C}_0(\Omega)$ denotes the set of continuous functions in $\mathcal{C}(\Omega)$ having compact support in Ω ;
- $\mathcal{C}^k(\Omega)$ denotes the set of functions defined on $\Omega \subset \mathbb{R}^N$ which have continuous derivatives up to the order k ;
- $\mathcal{C}_0^k(\Omega)$ denotes the set of functions in $\mathcal{C}^k(\Omega)$ having compact support;
- $\mathcal{C}_\#^1(\mathbb{R})$ denotes the set of functions f of class $\mathcal{C}^1(\Omega)$ except than on a discrete set $\Gamma \subset \mathbb{R}$ and such that, for any $\gamma \in \Gamma$, the limits

$$\lim_{x \rightarrow \gamma^\pm} f(x)$$

exist and are finite;

- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on $\Omega \subset \mathbb{R}^N$; $\mathcal{D}'(\Omega)$ denotes the topological dual of $\mathcal{D}(\Omega)$, namely the set of distributions on Ω ;
- if $A \subset \mathbb{R}^N$ is a set, then χ_A denotes the characteristic function of A ;
- for any $\xi \in (\mathbb{R}^N)^*$, $\rho \in \mathbb{R}^*$, we set $\mathfrak{B}_\rho(\xi) = \{x \in (\mathbb{R}^N)^* : |x - \xi| < \rho\}$;
- $\text{supp}(f) = \overline{\{x \in (\mathbb{R}^N)^* : f(x) \neq 0\}}$;
- $\text{mon}(x) = \{y \in (\mathbb{R}^N)^* : x \sim y\}$;

- $\text{gal}(x) = \{y \in (\mathbb{R}^N)^* : x \sim_f y\}$;
- $\forall^{a.e.} x \in X$ means "for almost every $x \in X$ ";
- if $a, b \in \mathbb{R}^*$, then
 - $[a, b]_{\mathbb{R}^*} = \{x \in \mathbb{R}^* : a \leq x \leq b\}$;
 - $(a, b)_{\mathbb{R}^*} = \{x \in \mathbb{R}^* : a < x < b\}$;
 - $]a, b[= [a, b]_{\mathbb{R}^*} \setminus (\text{mon}(a) \cup \text{mon}(b))$.

2 Definition of Ultrafunctions

In this section we introduce a few Desideratum that will be used to introduce ultrafunctions in a slightly different way with respect to what we did in [2], [3].

Let $\mathfrak{X} = \mathcal{P}_{fin}(\mathfrak{F}(\mathbb{R}, \mathbb{R}))$. Given $\lambda \in \mathfrak{X}$, we set $V_\lambda = \{\text{Span}(f_j) \mid f_j \in \lambda\}$.

Definition 1. *An internal function*

$$u = \lim_{\lambda \uparrow \Lambda} u_\lambda \in \mathfrak{F}(\mathbb{R})^*$$

is called *ultrafunction* if, for every $\lambda \in \mathfrak{X}$, $u_\lambda \in V_\lambda$. The space of ultrafunctions will be denoted by $\widetilde{\mathfrak{F}(\mathbb{R})}$. With some abuse of notation we will call ultrafunction also the restriction of u to any internal subset of \mathbb{R}^* .

In particular, we have that

$$\widetilde{\mathfrak{F}(\mathbb{R})} = \lim_{\lambda \uparrow \Lambda} V_\lambda,$$

so, being a Λ -limit of finite dimensional vector spaces, the vector space of ultrafunctions has hyperfinite dimension. Moreover, given any vector space of functions $W \subset \mathfrak{F}(\mathbb{R})$, we can define the space of ultrafunctions generated by W as follows:

$$\widetilde{W} = W^* \cap \widetilde{\mathfrak{F}(\mathbb{R})}.$$

Let us observe that

$$\widetilde{W} = \lim_{\lambda \uparrow \Lambda} W_\lambda,$$

where for every $\lambda \in \mathfrak{X}$ we pose $W_\lambda = V_\lambda \cap W$.

The space of ultrafunctions $\widetilde{\mathfrak{F}(\mathbb{R})}$ is too large for applications. We want to have a smaller space $\mathfrak{U}(\mathbb{R}) \subset \widetilde{\mathfrak{F}(\mathbb{R})}$ which satisfies suitable properties for applications. We list the main properties that we would like to obtain for $\mathfrak{U}(\mathbb{R})$.

Desideratum 1. *There is an infinite number β such that if $u(x) \in \mathfrak{U}(\mathbb{R})$, then $u(x) = 0$ for $|x| > \beta$ and $u(x) \in L^\infty(\mathbb{R})^*$.*

Desideratum 1 states that the ultrafunctions have an uniform compact support and are bounded in \mathbb{R}^* . From these conditions it follows that, if $u(x) \in \mathfrak{U}(\mathbb{R})$, then $u(x) \in L^p(\mathbb{R})^*$ for every p ; in particular, $u(x)$ is summable and it is in $L^2(\mathbb{R})^*$. So $\mathfrak{U}(\mathbb{R}) \subset L^2(\mathbb{R})^*$, and this allows to give to $\mathfrak{U}(\mathbb{R})$ the euclidean structure and the norm induced by $L^2(\mathbb{R})^*$.

Desideratum 2. $\mathfrak{U}(\mathbb{R}) \subset F_\#(\mathbb{R})^*$, where

$$F_\#(\mathbb{R}) = \left\{ u \in L_{loc}^1 \mid u(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} u(y) dy \right\}.$$

This request, which may seem strange at first sight, will allow to associate to every point $a \in [-\beta, \beta]$ a delta (or Dirac) ultrafunction centered in a , namely an ultrafunction $\delta_a(x)$ such that, for every ultrafunction $u(x)$, we have

$$\int^* u(x) \delta_a(x) dx = u(a).$$

Desideratum 3. If $f \in \mathcal{C}^1(\mathbb{R})$, and $a, b \in \mathbb{R}$, then $(f \cdot \chi_{[a,b]})^* \in \mathfrak{U}(\mathbb{R})$.

Desideratum 3 is introduced for a few different reasons. First of all, it is important to have the characteristic functions of intervals even if, due to Desideratum 2, we will have to pay attention in choosing the right definition of characteristic functions; moreover, it is important to have the extensions of \mathcal{C}^1 functions in $\mathfrak{U}(\mathbb{R})$ (one could ask this property for continuous function but, as we will show later, this request seems difficult to obtain if we want also the other Desideratums that we are presenting here). Finally, we will show that from Desideratum 3 it follows that the delta functions have compact support concentrated around their center: in fact we will show that, $\forall a \in \mathbf{gal}(0)$, $\mathbf{supp}(\delta_a) \subset \mathbf{mon}(a)$.

However it would be nice to have the previous property in the following more general fashion:

Desideratum 4. $\forall a \in [-\beta, \beta]$, $\mathbf{supp}(\delta_a) \subset \mathbf{mon}(a)$.

Our next desideratum is the following:

Desideratum 5. There exists a linear map $\widetilde{(\cdot)} : [L_{loc}^1(\Omega)]^* \rightarrow \mathfrak{U}(\mathbb{R})$ such that $\forall f \in [L_{loc}^1(\Omega)]^*$, $\forall v \in \mathfrak{U}(\mathbb{R})$, we have

$$\int^* f v dx = \int^* \widetilde{f} v dx.$$

Desideratum 5 substantially states that it is possible to define the projection of an $[L_{loc}^1(\Omega)]^*$ function on $\mathfrak{U}(\mathbb{R})$. In particular, this is useful to associate canonically an ultrafunction to every function $f \in L_{loc}^1(\Omega)$ since, in general, it will be false that $f^* \in \mathfrak{U}(\mathbb{R})$ (but when $f^* \in \mathfrak{U}(\mathbb{R})$ by Desideratum 5 we have $f^* = \widetilde{f}$).

Desideratum 6. *There exists a map $D : \mathfrak{U}(\mathbb{R}) \rightarrow \mathfrak{U}(\mathbb{R})$ such that*

- $\forall f \in \mathcal{C}^1(\mathbb{R}), \forall x \in \mathbb{R}, D\tilde{f}(x) = \tilde{f}'(x);$
- $\forall u, v \in \mathfrak{U}(\mathbb{R}), \int_{-\beta}^{\beta} Du(x)v(x)dx = - \int_{-\beta}^{\beta} u(x)Dv(x)dx + [u(x)v(x)]_{-\beta}^{\beta};$
- $D\tilde{1} = 0;$
- $D\chi_{[a,b]} = \delta_a - \delta_b.$

Desideratum 6 simply states that it is possible to define a derivative on $\mathfrak{U}(\mathbb{R})$ which satisfies a few expected properties.

In the next sections we show how to construct a space that satisfies all the Desideratum that we presented.

3 Construction of a canonical space of ultrafunctions

We want to consider a special subset of ultrafunctions. Let β be an infinite number; we set

$$\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_{\ell}\} \subset \mathbb{R}^*,$$

where $\gamma_0 = -\beta; \gamma_{\ell} = \beta$ and, for $j = 0, 1, \dots, \ell - 1$, we require that

$$0 < \gamma_{j+1} - \gamma_j < \eta$$

where η is an infinitesimal number. Moreover, it is useful to assume that $\mathbb{R} \subseteq \Gamma$.

For $j = 0, 1, \dots, \ell - 1$, we set

$$\mathbb{I}_j := (\gamma_j, \gamma_{j+1})_{\mathbb{R}^*}.$$

For every $a, b \in \Gamma$ we denote by $\chi_{[a,b]}(x)$ the characteristic function of $[a, b]$ defined in a slightly different way:

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in (a, b) \\ 0 & \text{if } x \notin [a, b] \\ \frac{1}{2} & \text{if } x = a, b; a \neq -\beta; b \neq \beta \\ 1 & \text{if } x = a = -\beta \\ 1 & \text{if } x = b = \beta \end{cases} \quad (1)$$

For every $j = 0, 1, \dots, \ell - 1$, we set

$$\chi_j(x) = \chi_{\mathbb{I}_j}(x).$$

The set of functions

$$\mathfrak{G} = \left\{ \sum_{j=0}^{\ell-1} c_j \chi_j(x) \mid c_j \in \mathbb{R}^* \right\}$$

will be referred to as the set of **grid functions**.

Definition 2. We denote by $\mathfrak{U}(\mathbb{R})$ the space of ultrafunctions

$$u : [-\beta, \beta] \rightarrow \mathbb{R}^*$$

which can be represented as follows:

$$u(x) = \sum_{j=0}^{\ell-1} v_j(x) \chi_j(x)$$

where, $\forall j \in J$, $v_j(x) \in \widetilde{\mathcal{C}^1(\mathbb{R})}$. We will refer to $\mathfrak{U}(\mathbb{R})$ as the canonical space of ultrafunctions.

Proposition 3. The elements of $\mathfrak{U}(\mathbb{R})$ are restriction to $[-\beta, \beta]$ of ultrafunctions.

Proof. Let $u(x) = \sum_{j=0}^{\ell} v_j(x) \chi_j(x)$, let $\ell = \lim_{\lambda \uparrow \Lambda} \ell_\lambda$, $\chi_j(x) = \lim_{\lambda \uparrow \Lambda} \chi_{j,\lambda}(x)$ and $v_j(x) = \lim_{\lambda \uparrow \Lambda} v_{j,\lambda}$. Then

$$u(x) = \lim_{\lambda \uparrow \Lambda} \sum_{j=0}^{\ell_\lambda-1} v_{j,\lambda}(x) \chi_{j,\lambda}(x),$$

so it is an ultrafunction. \square

Proposition 4. $\mathfrak{U}(\mathbb{R})$ is an hyperfinite dimensional vector space, and $\dim(\mathfrak{U}(\mathbb{R})) \leq \ell \cdot \dim \widetilde{\mathcal{C}^1(\mathbb{R})}$.

Proof. If $B = \{v_i(x) \mid i \leq \dim(\widetilde{\mathcal{C}^1(\mathbb{R})})\}$ is a basis for $\widetilde{\mathcal{C}^1(\mathbb{R})}$, the set

$$B_V = \{v_i(x) \chi_j(x) \mid v_i \in B, j = 0, \dots, \ell\}$$

is a set of generators for $\mathfrak{U}(\mathbb{R})$, and its cardinality is $\ell \cdot \dim \widetilde{\mathcal{C}^1(\mathbb{R})}$. So $\dim(\mathfrak{U}(\mathbb{R})) \leq \ell \cdot \dim \widetilde{\mathcal{C}^1(\mathbb{R})}$. \square

Since $\mathfrak{U}(\mathbb{R}) \subset [L^2(\mathbb{R})]^*$, it can be equipped with the following scalar product

$$(u, v) = \int^* u(x) \overline{v(x)} dx,$$

where \int^* is the natural extension of the Lebesgue integral considered as a functional

$$\int : L^1(\Omega) \rightarrow \mathbb{C}.$$

The norm of a (canonical) ultrafunction will be given by

$$\|u\| = \left(\int^* |u(x)|^2 dx \right)^{\frac{1}{2}}.$$

Canonical ultrafunctions have a few interesting properties:

Proposition 5. *The following properties hold:*

1. *If $f \in \mathcal{C}^1(\mathbb{R})$ then $f^* \cdot \chi_{[-\beta, \beta]_{\mathbb{R}^*}} \in \mathfrak{U}(\mathbb{R})$;*
2. *if $u \in \mathfrak{U}(\mathbb{R})$ and $a, b \in \Gamma$, then $u \cdot \chi_{[a, b]_{\mathbb{R}^*}} \in \mathfrak{U}(\mathbb{R})$;*
3. *if $u \in \mathfrak{U}(\mathbb{R})$ then for $j = 1, \dots, \ell - 1$ the limits*

$$\left(\lim_{x \rightarrow \gamma_j^\pm} \right)^* u(x)$$

are well defined and we set

$$u(\gamma_j^+) := \left(\lim_{x \rightarrow \gamma_j^+} \right)^* u(x); \quad u(\gamma_j^-) := \left(\lim_{x \rightarrow \gamma_j^-} \right)^* u(x); \quad (2)$$

4. *if $u \in \mathfrak{U}(\mathbb{R})$ then for $j = 0$ the limit*

$$\left(\lim_{x \rightarrow \gamma_0^+} \right)^* u(x)$$

is well defined and for $j = \ell$ the limit

$$\left(\lim_{x \rightarrow \gamma_\ell^-} \right)^* u(x)$$

is well defined.

5. *if, for every $j = 0, \dots, \ell$ we set*

$$V(\mathbb{I}_j) := \{u(x)\chi_j(x) \mid u(x) \in \widetilde{\mathcal{C}^1(\mathbb{R})}\},$$

then, for $k \neq j$, $V(\mathbb{I}_j)$ and $V(\mathbb{I}_k)$ are orthogonal;

6. *$\mathfrak{U}(\mathbb{R})$ can be splitted in orthogonal spaces as follows:*

$$\mathfrak{U}(\mathbb{R}) = \bigoplus_{j=0}^{\ell} V(\mathbb{I}_j).$$

Proof. 1) If $f \in \mathcal{C}^1(\mathbb{R})$, then $f^* \in \widetilde{\mathcal{C}^1(\mathbb{R})}$, and

$$f^* \cdot \chi_{[-\beta, \beta]_{\mathbb{R}^*}} = \sum_{j=0}^{\ell-1} f^*(x)\chi_j(x) \in \mathfrak{U}(\mathbb{R}).$$

- 2) It follows by (1).

3) If $u(x) = \sum_{j=0}^{\ell-1} u_j(x)\chi_j(x)$, then

$$u(\gamma_0^-) = \left(\lim_{x \rightarrow \gamma_j^-} \right)^* u_{j-1}(x)$$

and

$$u(\gamma_j^+) = \left(\lim_{x \rightarrow \gamma_j^+} \right)^* u_j(x)$$

and these limits exist because u_{j-1}, u_j are continuous on $\overline{\mathbb{I}_{j-1}}, \overline{\mathbb{I}_j}$ respectively.

4) The same as in 1).

5) This is immediate since, if $j \neq k$, if $u \in V(\mathbb{I}_j)$ and $v \in V(\mathbb{I}_k)$ then the supports of u and v are disjoint.

6) Having proved 3), it remains only to prove that $\bigoplus_{j=0}^{\ell} V(\mathbb{I}_j)$ generates all

$\mathfrak{U}(\mathbb{R})$, and this is clear because, if $u(x) = \sum_{j=0}^{\ell} u_j(x)\chi_j(x)$ then, for every $j = 0, \dots, \ell - 1$, $u_j(x)\chi_j(x) \in V(\mathbb{I}_j)$. \square

Definition 6. A basis $\{e_{j,k} : j = 0, \dots, \ell - 1, k = 1, \dots, s_j\}$ for $\mathfrak{U}(\mathbb{R})$ is called *splitted basis* if, for every $j = 0, \dots, \ell$, $\{e_{j,k}\}_{k=1}^{s_j}$ is a basis for $V(\mathbb{I}_j)$.

4 Delta and Sigma basis

Following the approach presented in [3], in this section we introduce two particular bases for $\mathfrak{U}(\mathbb{R})$ and we study their main properties. We start by defining the *Delta ultrafunctions*. In order to do this, it is useful to observe that the value of an ultrafunction u for γ_j , $j = 1, \dots, \ell - 1$, can be defined as follows:

$$u(\gamma_j) = \frac{u(\gamma_j^+) + u(\gamma_j^-)}{2}$$

where $u(x^+), u(x^-)$ are defined by (2). The fact that this definition makes sense follows by points 3) and 4) in Proposition 5. Moreover we pose

$$u(\gamma_0) = u(-\beta) = u^+(-\beta); \quad u(\gamma_\ell) = u(\beta) = u^-(-\beta).$$

These observations are relevant in the following definition:

Definition 7. Given a number $q \in [-\beta, \beta]$ we denote by $\delta_q(x)$ an ultrafunction in $\mathfrak{U}(\mathbb{R})$ such that

$$\forall v \in \mathfrak{U}(\mathbb{R}), \quad \int^* v(x)\delta_q(x)dx = v(q). \quad (3)$$

$\delta_q(x)$ is called the *Delta (or Dirac) ultrafunction concentrated in q* .

Let us see the main properties of the Delta ultrafunctions:

Theorem 8. *We have the following properties:*

1. For every $q \in [-\beta, \beta]$ there exists an unique Delta ultrafunction concentrated in q ;
2. for every $a, b \in [-\beta, \beta]$ $\delta_a(b) = \delta_b(a)$;
3. $\|\delta_q\|^2 = \delta_q(q)$.

Proof. 1) Let $\{e_{j,k} : j = 0, \dots, \ell - 1, k = 1, \dots, s_j\}$ be an orthogonal splitted basis of $\mathfrak{U}(\mathbb{R})$ (see Def. 6). If $q \in \mathbb{I}_j$ we pose

$$\delta_q(x) = \sum_{k=1}^{s_j} e_{j,k}(q) e_{j,k}(x).$$

For every $i \neq j$, for every $v \in V(\mathbb{I}_i)$ we have $\int^* v(x) \delta_q(x) dx = 0 = v(q)$. If $v \in V(\mathbb{I}_j)$, $v = \sum_{k=1}^{s_j} v_k e_{j,k}(x)$ we have

$$\begin{aligned} \int^* v(x) \delta_q(x) dx &= \\ \int^* \left(\sum_{k=1}^{s_j} e_{j,k}(q) e_{j,k}(x) \right) \left(\sum_{k=1}^{s_j} v_k e_{j,k}(x) \right) dx &= \sum_{k=1}^{s_j} \int^* e_{j,k}(q) e_{j,k}(x) v_k e_{j,k}(x) dx = \\ \sum_{k=1}^{s_j} e_{j,k}(q) v_k &= v(q). \end{aligned}$$

If $q = \gamma_0$ we pose

$$\delta_q(x) = \sum_{k=1}^{s_0} e_{j,k}^+(q) e_{j,k}(x)$$

and if $q = \gamma_\ell$ we pose

$$\delta_q(x) = \sum_{k=1}^{s_{\ell-1}} e_{j,k}^-(q) e_{j,k}(x).$$

The verification that these definitions are well posed is equal to the one carried out for $q \in \mathbb{I}_j$.

If $q = \gamma_j$, $j \neq 0, \ell$ we set

$$\delta_q(x) = \frac{1}{2} \left(\sum_{k=1}^{s_{j-1}} e_{j-1,k}^-(q) e_{j-1,k}(x) + \sum_{k=1}^{s_j} e_{j,k}^+(q) e_{j,k}(x) \right).$$

Then

$$\int^* v(x) \delta_q(x) dx =$$

$$\begin{aligned}
& \frac{1}{2} \left(\int_{[\gamma_{j-1}, \gamma_j]}^* v(x) \left(\sum_{k=1}^{s_j} e_{j-1,k}^-(q) e_{j-1,k}(x) \right) dx + \int_{[\gamma_j, \gamma_{j+1}]}^* v(x) \left(\sum_{k=1}^{s_j} e_{j,k}^+(q) e_{j,k}(x) \right) dx \right) = \\
& \frac{1}{2} \left(\int_{[\gamma_{j-1}, \gamma_j]}^* v^-(x) \left(\sum_{k=1}^{s_j} e_{j-1,k}^-(q) e_{j-1,k}(x) \right) dx + \int_{[\gamma_j, \gamma_{j+1}]}^* v^+(x) \left(\sum_{k=1}^{s_j} e_{j,k}^+(q) e_{j,k}(x) \right) dx \right) = \\
& \frac{1}{2} [v^-(\gamma_j) + v^+(\gamma_j)] = v(\gamma_j).
\end{aligned}$$

The Delta function in q is unique: if $f_q(x)$ is another Delta ultrafunction centered in q then for every $y \in [-\beta, \beta]$ we have:

$$\delta_q(y) - f_q(y) = \int^* (\delta_q(x) - f_q(x)) \delta_y(x) dx = \delta_y(q) - \delta_y(q) = 0$$

and hence $\delta_q(y) = f_q(y)$ for every $y \in (-\beta, \beta)$.

2. $\delta_a(b) = \int^* \delta_a(x) \delta_b(x) dx = \delta_b(a)$.

3. $\|\delta_q\|^2 = \int^* \delta_q(x) \delta_q(x) dx = \delta_q(q)$. □

Let us observe that, as the previous proof shows, in every point γ_j of the grid Γ , with the exceptions of $-\beta, \beta$, it is possible to define three delta functions centered in γ_j , namely $\delta_{\gamma_j}^-(x)$, $\delta_{\gamma_j}^+(x)$ and $\delta_{\gamma_j}(x)$, which satisfy the following properties: for every $\forall v \in \mathfrak{U}(\mathbb{R})$, we have

$$\begin{aligned}
\int^* v(x) \delta_{\gamma_j}^-(x) dx &= v^-(\gamma_j); \\
\int^* v(x) \delta_{\gamma_j}^+(x) dx &= v^+(\gamma_j); \\
\int^* v(x) \delta_{\gamma_j}(x) dx &= v(\gamma_j).
\end{aligned} \tag{4}$$

Moreover, it is immediate to prove that the conditions in (4) characterize uniquely the functions $\delta_{\gamma_j}^-(x)$, $\delta_{\gamma_j}^+(x)$ and $\delta_{\gamma_j}(x)$. So we will consider (4) as a definition for $\delta_{\gamma_j}^-(x)$, $\delta_{\gamma_j}^+(x)$ and $\delta_{\gamma_j}(x)$.

Definition 9. A Delta-basis $\{\delta_a(x)\}_{a \in \Sigma}$ ($\Sigma \subset [-\beta, \beta]$) is a basis for $\mathfrak{U}(\mathbb{R})$ whose elements are Delta ultrafunctions. Its dual basis $\{\sigma_a(x)\}_{a \in \Sigma}$ is called Sigma-basis. We recall that, by definition of dual basis, for every $a, b \in \Sigma$ the equation

$$\int^* \delta_a(x) \sigma_b(x) dx = \delta_{ab} \tag{5}$$

holds. A set $A \subset [-\beta, \beta]$ is called set of independent points if $\{\delta_a(x)\}_{a \in A}$ is a basis

The existence of a Delta-basis is an immediate consequence of the following fact:

Remark 10. The set $\{\delta_a(x) | a \in [-\beta, \beta]\}$ generates all $\mathfrak{U}(\mathbb{R})$. In fact, let $G(\Omega)$ be the vector space generated by the set $\{\delta_a(x) | a \in [-\beta, \beta]\}$ and suppose that $G(\Omega)$ is properly included in $\mathfrak{U}(\mathbb{R})$. Then the orthogonal $G(\Omega)^\perp$ of $G(\Omega)$ in $\mathfrak{U}(\mathbb{R})$ contains a function $f \neq 0$. But, since $f \in G(\Omega)^\perp$, for every $a \in [-\beta, \beta]$ we have

$$f(a) = \int^* f(x) \delta_a(x) dx = 0,$$

so $f|_{[-\beta, \beta]} = 0$ and this is absurd. Thus the set $\{\delta_a(x) | a \in (-\beta, \beta)\}$ generates $\mathfrak{U}(\mathbb{R})$, hence it contains a basis.

Let us see some properties of Delta- and Sigma-bases (which, in this new context, are slightly different from the one presented in [3]):

Theorem 11. A Delta-basis $\{\delta_q(x)\}_{q \in \Sigma}$ and its dual basis $\{\sigma_q(x)\}_{q \in \Sigma}$ satisfy the following properties:

1. if $u \in \mathfrak{U}(\mathbb{R})$ then

$$u(x) = \sum_{q \in \Sigma} \left(\int^* \sigma_q(\xi) u(\xi) d\xi \right) \delta_q(x);$$

2. if $u \in \mathfrak{U}(\mathbb{R})$ then

$$u(x) = \sum_{q \in \Sigma} u(q) \sigma_q(x); \quad (6)$$

3. if two ultrafunctions u and v coincide on a set of independent points then they are equal;

4. if Σ is a set of independent points and $a, b \in \Sigma$ then $\sigma_a(b) = \delta_{ab}$;

5. for every $q \in [-\beta, \beta]$, $\sigma_q(x)$ is well defined;

6. for every $q \in [-\beta, \beta]$ if $q \in \mathbb{I}_j$ then $\text{supp}(\delta_q(x)) \subset \overline{\mathbb{I}_j}$ and $\text{supp}(\sigma_q(x)) \subset \overline{\mathbb{I}_j}$;

7. for every $\gamma_j \in \Gamma \setminus \{\gamma_0, \gamma_\ell\}$, $\text{supp}(\delta_{\gamma_j}(x)) \subset \overline{\mathbb{I}_{j-1} \cup \mathbb{I}_j}$ and $\text{supp}(\sigma_{\gamma_j}(x)) \subset \overline{\mathbb{I}_{j-1} \cup \mathbb{I}_j}$;

8. $\text{supp}(\delta_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\text{supp}(\sigma_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\text{supp}(\delta_{\gamma_\ell}(x)) \subset \overline{\mathbb{I}_\ell}$ and $\text{supp}(\sigma_{\gamma_\ell}(x)) \subset \overline{\mathbb{I}_{\ell-1}}$;

9. for every $q \in [-\beta, \beta]$, $\text{supp}(\delta_q(x)) \subset \text{mon}(q)$ and $\text{supp}(\sigma_q(x)) \subset \text{mon}(q)$.

Proof. 1) It is an immediate consequence of the definition of dual basis.

2) Since $\{\delta_q(x)\}_{q \in \Sigma}$ is the dual basis of $\{\sigma_q(x)\}_{q \in \Sigma}$ we have that

$$u(x) = \sum_{q \in \Sigma} \left(\int \delta_q(\xi) u(\xi) d\xi \right) \sigma_q(x) = \sum_{q \in \Sigma} u(q) \sigma_q(x).$$

- 3) It follows directly from the previous point.
 4) It follows directly by equation (5).
 5) Given any point $q \in (-\beta, \beta)$ clearly there is a Delta-basis $\{\delta_a(x)\}_{a \in \Sigma}$ with $q \in \Sigma$. Then $\sigma_q(x)$ can be defined by mean of the basis $\{\delta_a(x)\}_{a \in \Sigma}$. We have to prove that, given another Delta basis $\{\delta_a(x)\}_{a \in \Sigma'}$ with $q \in \Sigma'$, the corresponding $\sigma'_q(x)$ is equal to $\sigma_q(x)$. Using (2), with $u(x) = \sigma'_q(x)$, we have that

$$\sigma'_q(x) = \sum_{a \in \Sigma} \sigma'_q(a) \sigma_a(x).$$

Then, by (4), it follows that $\sigma'_q(x) = \sigma_q(x)$.

6) As we proved in Theorem 8, if $q \in \mathbb{I}_j$ then δ_q is an element of $V(\mathbb{I}_j)$, so $\text{supp}(\delta_q(x)) \subset \overline{\mathbb{I}_j}$. Now $\delta_q \in V(\mathbb{I}_j)$, so there is a corrispective function $\sigma_q \in V(\mathbb{I}_j)$ which is the sigma function centered in q . If we extend this function to $[-\beta, \beta]$ by posing $\sigma_q(x) = 0$ for $x \notin \mathbb{I}_j$ we obtain, by uniqueness, exactly the sigma function centered in q in $\mathfrak{U}(\mathbb{R})$. And $\text{supp}(\sigma_q(x)) \subset \overline{\mathbb{I}_j}$.

7) In Theorem 8 we proved that δ_{γ_j} is an element in $V(\mathbb{I}_j) \cup V(\mathbb{I}_{j+1})$, so $\text{supp}(\delta_{\gamma_j}(x)) \subset \overline{\mathbb{I}_{j-1} \cup \mathbb{I}_j}$. Now we can consider its corrispective sigma function $\sigma_{\gamma_j} \in V(\mathbb{I}_j) \cup V(\mathbb{I}_{j+1})$. If we extend this function to $\mathfrak{U}(\mathbb{R})$ by posing $\sigma_{\gamma_j}(x) = 0$ for $x \notin \mathbb{I}_j \cup \mathbb{I}_{j+1}$, we obtain the sigma function in $\mathfrak{U}(\mathbb{R})$ centered in γ_j . And, by construction, $\text{supp}(\sigma_{\gamma_j}(x)) \subset \overline{\mathbb{I}_j \cup \mathbb{I}_{j+1}}$.

8) In Theorem 8 we proved that δ_0 is an element in $V(\mathbb{I}_0)$ and δ_ℓ is in $V(\mathbb{I}_{\ell-1})$, and that the same property holds for the corrispective σ functions can be proved as in point (6) of this Theorem. So $\text{supp}(\delta_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\text{supp}(\sigma_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\text{supp}(\delta_{\gamma_{\ell+1}}(x)) \subset \overline{\mathbb{I}_\ell}$ and $\text{supp}(\sigma_{\gamma_\ell}(x)) \subset \overline{\mathbb{I}_{\ell-1}}$.

9) It is a straightforward consequence of the points 6 and 7, since for every $j \in J$ we have $\mathbb{I}_j \cup \mathbb{I}_{j+1} \subset \text{mon}(q)$. \square

5 Canonical extension of functions

We start by defining a map

$$\widetilde{(\cdot)} : [L^1_{loc}(\mathbb{R})]^* \rightarrow \mathfrak{U}(\mathbb{R})$$

which will be very useful in the extension of functions.

Definition 12. If $u \in [L^1_{loc}(\mathbb{R})]^*$, \tilde{u} denotes the unique ultrafunction such that

$$\forall v \in \mathfrak{U}(\mathbb{R}), \quad \int^* \tilde{u}(x)v(x)dx = \int^* u(x)v(x)dx.$$

Remark 13. Notice that, if $u \in [L^2(\mathbb{R})]^*$, then $\tilde{u} = P_V u$ where

$$P_V : [L^2(\mathbb{R})]^* \rightarrow \mathfrak{U}(\mathbb{R})$$

is the orthogonal projection.

The following theorem shows that \tilde{u} is well defined and unique.

Theorem 14. *If $u \in [L^1_{loc}(\mathbb{R})]^*$ then*

$$\tilde{u}(x) = \sum_{q \in \Sigma} \left[\int u(\xi) \delta_q(\xi) d\xi \right] \sigma_q(x) \quad (7)$$

$$= \sum_{q \in \Sigma} \left[\int u(\xi) \sigma_q(\xi) d\xi \right] \delta_q(x). \quad (8)$$

Proof. It is sufficient to prove that

$$\forall v \in \mathfrak{U}(\mathbb{R}), \quad \int \sum_{q \in \Sigma} \left[\int u(\xi) \delta_q(\xi) d\xi \right] \sigma_q(x) v(x) dx = \int u(\xi) v(\xi) d\xi.$$

We have that $v(x) = \sum_{q \in \Sigma} v_q \delta_q(x)$ with $v_q = \int \sigma_q(x) v(x) dx$; then

$$\begin{aligned} \int \sum_{q \in \Sigma} \left[\int u(\xi) \delta_q(\xi) d\xi \right] \sigma_q(x) v(x) dx &= \sum_{q \in \Sigma} \left(\int u(\xi) \delta_q(\xi) d\xi \right) \left(\int \sigma_q(x) v(x) dx \right) = \\ &= \sum_{q \in \Sigma} \left(\int u(\xi) \delta_q(\xi) d\xi \right) v_q = \int u(\xi) \left[\sum_{q \in \Sigma} v_q \delta_q(\xi) \right] d\xi = \int u(\xi) v(\xi) d\xi. \end{aligned}$$

The other equalities can be proved similarly. \square

In particular, if $f \in L^1_{loc}(\mathbb{R})$, the function $\widetilde{f^*}$ is well defined. From now on we will simplify the notation just writing \widetilde{f} .

Example 15. *Take $|x|^{-1/2} \in L^1_{loc}(-1, 1)$, then*

$$\widetilde{|x|^{-1/2}} = \sum_{q \in \Sigma} \left(\int^* |\xi|^{-1/2} \delta_q(\xi) d\xi \right) \sigma_q(x)$$

makes sense for every $x \in \mathbb{R}^$; in particular*

$$\left(\widetilde{|x|^{-1/2}} \right)_{x=0} = \int^* |x|^{-1/2} \delta_0(x) dx,$$

and it is easy to check that this is an infinite number. Notice that the ultrafunction $\widetilde{|x|^{-1/2}}$ is different from $(|x|^{-1/2})^$ since the latter is not defined for $x = 0$ (and they also differ for $|x| > \beta$).*

Now we want to show some interesting relations between \widetilde{f} and f^* . More precisely we are interested in the following question.

Take $f \in L^1_{loc}(\mathbb{R})$ and $\Omega \subset \mathbb{R}$; which are the conditions that ensure the following:

$$\forall x \in \Omega^*, \quad \widetilde{f}(x) = f^*(x)? \quad (\text{Q})$$

Notice that if $f \in L^1_{loc}(\mathbb{R})$, f and f^* are not defined pointwise and hence the above equality must be intended for almost every x .

Lemma 16. *Let $\Omega \subset \mathbb{R}$ be an open set and let $f \in L^1_{loc}(\mathbb{R})$. Then*

$$\forall^{a.e.} x \in \Omega \quad f(x) = 0 \Leftrightarrow \forall x \in \Omega^* \cap [-\beta, \beta] \quad \tilde{f}(x) = 0.$$

Proof. We recall that, by (7),

$$\tilde{f}(x) = \sum_{q \in \Sigma} \left[\int f^*(\xi) \delta_q(\xi) d\xi \right] \sigma_q(x).$$

If $\forall^{a.e.} x \in \Omega$, $f(x) = 0$ then by Leibnitz principle we have that $\forall^{a.e.} x \in \Omega^*$, $f^*(x) = 0$, so that $\forall x \in \Omega^* \cap [-\beta, \beta]$, $\tilde{f}(x) = 0$ follows by (7). Conversely, let us suppose that there is an open bounded interval $I \subseteq \Omega$ such that $\forall^{a.e.} x \in I$ $f(x) \neq 0$ (we suppose that $\forall^{a.e.} x \in I$ $f(x) > 0$). By Leibnitz principle, we have that $\forall^{a.e.} x \in I^*$ $f^*(x) > 0$. Let $q \in \mathbb{I}_j$. Then

$$\begin{aligned} 0 &< \int_{\mathbb{I}_j}^* f^*(x) \chi_j(x) dx = \int^* f^*(x) \chi_j(x) dx = \\ &= \int^* \tilde{f}(x) \chi_j(x) dx = 0, \end{aligned}$$

since by hypothesis $\tilde{f}(x) = 0 \quad \forall x \in \Omega^* \cap [-\beta, \beta] \supset I^*$. And this is clearly absurd. \square

Corollary 17. *Let $\Omega \subset \mathbb{R}$ be an open set and let $f, g \in L^1_{loc}(\mathbb{R})$; then*

$$\forall^{a.e.} x \in \Omega \quad f(x) = g(x) \Leftrightarrow \forall x \in \Omega^* \cap [-\beta, \beta] \quad \tilde{f}(x) = \tilde{g}(x).$$

Proof. This follows immediatly by applying the previous theorem to the function $h(x) = f(x) - g(x)$, since the operation $f \rightarrow \tilde{f}$ is linear. \square

Theorem 18. *Let $\Omega \subset \mathbb{R}$ be an open bounded set, let $f \in L^1_{loc}(\mathbb{R})$; if $f|_{\Omega} \in \mathcal{C}^1(\Omega)$ then*

$$\forall x \in \Omega^* \cap [-\beta, \beta] \quad \tilde{f}(x) = f^*(x).$$

Proof. Let $\{\delta_a(x)\}_{a \in \Sigma}$ be a Delta basis, let $y \in \Omega^*$ and let $y \in \mathbb{I}_j$. Since, by (11), for every $q \in \Sigma$ with $q \notin \mathbb{I}_j$ $\sigma_q(y) = 0$, by (7) we deduce that

$$\tilde{f}(y) = \sum_{q \in \Sigma \cap \mathbb{I}_j} \left[\int_{\mathbb{I}_j} f^*(\xi) \delta_q(\xi) d\xi \right] \sigma_q(y).$$

Now let $g_j(x)$ be the function such that

$$g_j(x) = \begin{cases} f^*(x) & \text{if } x \in \mathbb{I}_j; \\ 0 & \text{otherwise.} \end{cases}$$

Since $f|_{\Omega} \in \mathcal{C}^1(\Omega)$ then $g_j(x)$ is an ultrafunction. By construction, we have that $g_j(y) = \tilde{f}(y)$ since, by (6),

$$\begin{aligned} g_j(y) &= \sum_{q \in \Sigma} g_j(q) \sigma_q(y) = \sum_{q \in \Sigma \cap \mathbb{I}_j} \left[\int_{\mathbb{I}_j} g_j(\xi) \delta_q(\xi) d\xi \right] \sigma_q(y) = \\ &= \sum_{q \in \Sigma \cap \mathbb{I}_j} \left[\int_{\mathbb{I}_j} f^*(\xi) \delta_q(\xi) d\xi \right] \sigma_q(y) = \tilde{f}(y). \end{aligned}$$

But, by definition, $g_j(y) = f^*(y)$; hence we deduce that $f^*(y) = \tilde{f}(y)$. \square

Example 19. If $f(x) = 1$, then

$$\tilde{1}(x) = \begin{cases} 1 & \text{if } x \in [-\beta, \beta]_{\mathbb{R}^*}; \\ 0 & \text{if } x \notin [-\beta, \beta]_{\mathbb{R}^*}. \end{cases}$$

By Theorem 18 and the above example, we get:

Corollary 20. Let $f \in \mathcal{C}^1(\Omega)$; then,

$$\tilde{f} = f^* \cdot \tilde{1}$$

By Theorem 18, given a function $f(x) \in \mathcal{C}^1(\mathbb{R})$ we have that $\tilde{f}(x)$ extends $f(x)$ to $[-\beta, \beta]_{\mathbb{R}^*}$. $\tilde{f}(x)$ will be called the **canonical extension of $f(x)$** . With some abuse of notation, $\tilde{f}(x)$ will be called the "canonical extension of $f(x)$ " even when $f(x) \in L^1_{loc}(\mathbb{R})$.

Example 21. If we consider the Example 15, by Theorem 18, we have that

$$\forall a \in [-\beta, \beta] \setminus \text{mon}(0), \quad \left(\widetilde{|x|^{-1/2}} \right)_{x=a} = \left(|x|^{-1/2} \right)_{x=a}^* = |a|^{-1/2}.$$

Example 22. For a fixed $k \in \mathbb{R}$, the function e^{ikx} defines a unique ultrafunction $\widetilde{e^{ikx}}$. Notice that $\widetilde{e^{ikx}}$ is different from the natural extension of e^{ikx} even if

$$\forall x \in \text{gal}(0), \quad \widetilde{e^{ikx}} = e^{ikx}.$$

6 Derivative

Definition 23. For every ultrafunction $u \in \mathfrak{U}(\mathbb{R})$, the derivative $Du(x)$ of $u(x)$ is the ultrafunction defined by the following formula:

$$Du(x) = P_{\mathfrak{U}} u' + \sum_{j=1}^{\ell-1} \Delta u(\gamma_j) \delta_{\gamma_j}(x), \quad (9)$$

where $P_{\mathfrak{U}} u'$ denotes the orthogonal projection of u' on $\mathfrak{U}(\mathbb{R})$ w.r.t. the L^2 scalar product and, for every $j = 1, \dots, \ell-1$,

$$\Delta u(\gamma_j) = u^+(\gamma_j) - u^-(\gamma_j).$$

Theorem 24. For every $u, v \in \mathfrak{U}(\mathbb{R})$ the following equality holds:

$$\int Du(x)v(x) \, dx = - \int u(x)Dv(x) \, dx + [u(x)v(x)]_{-\beta}^{\beta}. \quad (10)$$

Proof. We have:

$$\begin{aligned} & \int (Du(x)v(x) + u(x)Dv(x))dx = \\ & \int \left(P_V u'(x) + \sum_{j=1}^{\ell-1} \Delta u(\gamma_j) \delta_{\gamma_j}(x) \right) v(x) dx + \int \left(P_V v'(x) + \sum_{j=1}^{\ell-1} \Delta v(\gamma_j) \delta_{\gamma_j}(x) \right) u(x) dx = \\ & \int [P_V u'(x)v(x) + u(x)P_V v'(x)] \, dx + \\ & \int \left[\left(\sum_{j=0}^{\ell-1} \Delta u(\gamma_j) \delta_{\gamma_j}(x) \right) v(x) + \left(\sum_{j=0}^{\ell-1} \Delta v(\gamma_j) \delta_{\gamma_j}(x) \right) u(x) \right] \, dx = \\ & \int [P_V u'(x)v(x) + u(x)P_V v'(x)] \, dx + \sum_{j=1}^{\ell-1} [\Delta u(\gamma_j)v(\gamma_j) + \Delta v(\gamma_j)u(\gamma_j)]. \end{aligned}$$

Now let us compute the two terms of the sum separately; the first one:

$$\begin{aligned} \int [P_V u'(x)v(x) + u(x)P_V v'(x)] \, dx &= \sum_{j=0}^{\ell-1} \int_{\gamma_j}^{\gamma_{j+1}} [P_V u'(x)v(x) + u(x)P_V v'(x)] \, dx = \\ & \sum_{j=0}^{\ell-1} \int_{\gamma_j}^{\gamma_{j+1}} [u'(x)v(x) + u(x)v'(x)] \, dx = \sum_{j=0}^{\ell-1} \int_{\gamma_j}^{\gamma_{j+1}} (u(x)v(x))' \, dx = \\ &= \sum_{j=0}^{\ell-1} [u^-(\gamma_{j+1})v^-(\gamma_{j+1}) - u^+(\gamma_j)v^+(\gamma_j)]. \end{aligned}$$

The second one:

$$\begin{aligned} & \sum_{j=1}^{\ell-1} [\Delta u(\gamma_j)v(\gamma_j) + \Delta v(\gamma_j)u(\gamma_j)] = \\ & \sum_{j=1}^{\ell-1} \left((u^+(\gamma_j) - u^-(\gamma_j)) \left(\frac{v^+(\gamma_j) + v^-(\gamma_j)}{2} \right) + (v^+(\gamma_j) - v^-(\gamma_j)) \left(\frac{u^+(\gamma_j) + u^-(\gamma_j)}{2} \right) \right) = \\ & \sum_{j=1}^{\ell-1} (u^+(\gamma_j)v^+(\gamma_j) - u^-(\gamma_j)v^-(\gamma_j)). \end{aligned}$$

Thus

$$\begin{aligned}
& \int [P_V u' v(x) + u(x) P_V v'] dx + \sum_{j=0}^{\ell-1} (\Delta u(\gamma_j) v(\gamma_j) + \Delta v(\gamma_j) u(\gamma_j)) = \\
& \sum_{j=0}^{\ell-1} (u^-(\gamma_{j+1}) v^-(\gamma_{j+1}) - u^+(\gamma_j) v^+(\gamma_j)) + \sum_{j=1}^{\ell-1} (u^+(\gamma_j) v^+(\gamma_j) - u^-(\gamma_j) v^-(\gamma_j)). \\
& \text{But } \sum_{j=0}^{\ell-1} (u^-(\gamma_{j+1}) v^-(\gamma_{j+1}) - u^+(\gamma_j) v^+(\gamma_j)) = -u(-\beta) v(-\beta) + u(\beta) v(\beta) + \\
& \sum_{j=1}^{\ell-1} (u^-(\gamma_j) v^-(\gamma_j) - u^+(\gamma_j) v^+(\gamma_j)), \text{ hence} \\
& \sum_{j=0}^{\ell-1} (u^-(\gamma_{j+1}) v^-(\gamma_{j+1}) - u^+(\gamma_j) v^+(\gamma_j)) + \sum_{j=1}^{\ell-1} (u^+(\gamma_j) v^+(\gamma_j) - u^-(\gamma_j) v^-(\gamma_j)) = \\
& u(\beta) v(\beta) - u(-\beta) v(-\beta). \quad \square
\end{aligned}$$

Remark 25. *The generalized derivative*

$$D : \mathfrak{U}(\mathbb{R}) \rightarrow \mathfrak{U}(\mathbb{R})$$

is a linear operator, as can be directly derived by (9). Moreover for every ultrafunction $u \in \mathfrak{U}(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})^*$ we have that

$$Du(x) = \widetilde{u'(x)}, \quad (11)$$

since in this case $\Delta u(\gamma_j) = 0$ for every $j = 1, \dots, l-1$. In particular, if $f \in \mathcal{C}^2(\mathbb{R})$ then $\forall x \in [-\beta, \beta]$

$$Df^*(x) = (f')^*(x), \quad (12)$$

because in this case $(f')^*(x) \in \mathfrak{U}(\mathbb{R})$, so $P_{\mathfrak{U}}(f')^* = (f')^*$.

Remark 26. Notice that by (11) and (12) we have that $\forall f \in \mathcal{C}^1(\mathbb{R})$ and $\forall x \in \mathbb{R}$

$$D\widetilde{f}(x) = \widetilde{f'(x)} \sim f'(x)$$

and $\forall f \in \mathcal{C}^2(\mathbb{R})$ and $\forall x \in \mathbb{R}$

$$D\widetilde{f}(x) = f'(x).$$

In this sense, D extends the usual derivative to all ultrafunctions and to all the points in \mathbb{R}^* .

Example 1: By (10) we have that

$$D\tilde{1} = 0. \quad (13)$$

If $u(x) = \tilde{x}$ then

$$D\tilde{x} = \tilde{1}.$$

Example 2: If $a \neq -\beta, b \neq \beta$ and $u(x) = \chi_{[a,b]}(x)$, then

$$D\chi_{[a,b]} = \delta_a - \delta_b.$$

Example 3: If $a = -\beta, b \neq \beta$ and $u(x) = \chi_{[a,b]}(x)$, then

$$D\chi_{[a,b]} = -\delta_b,$$

and if $a \neq -\beta, b = \beta$ and $u(x) = \chi_{[a,b]}(x)$, then

$$D\chi_{[a,b]} = \delta_a.$$

Example 4: $u(x) = w(x)\chi_{[a,b]}(x)$ with $a, b \in \Gamma \setminus \{-\beta, \beta\}$, then, by (9)

$$u(x)' = P_V w'(x)\chi_{[a,b]}(x) + w(a)\delta_a(x) - w(b)\delta_b(x).$$

7 Definite integral

Since every ultrafunction is an internal function, the definite integral is well defined:

$$\int_a^b u(x)dx := \left(\int_a^b \right)^* u(x)dx.$$

Let us observe that, for every $a, b \in \Gamma$, the characteristic function $\chi_{[a,b]}$ of $[a, b]$ in the usual sense and the characteristic function $\chi_{[a,b]_{\mathbb{R}^*}}$ of $[a, b]$ in the sense of ultrafunctions are different (at most) only in the points a and b . In particular, for every ultrafunction $u(x)$ we have

$$\int_a^b u(x)dx = \int^* u(x)\chi_{[a,b]}(x)dx = \int^* u(x)\chi_{[a,b]_{\mathbb{R}^*}}(x)dx.$$

This observation is important to prove the following theorem:

Corollary 27. (*Fundamental Theorem of Calculus*) *If $a, b \in \Gamma$, then*

$$\int_a^b Du(x)dx = u(b) - u(a).$$

Proof. We have:

$$\begin{aligned} \int_a^b Du(x)dx &= \int^* Du(x)\chi_{[a,b]}(x)dx = \\ \int^* Du(x)\chi_{[a,b]_{\mathbb{R}^*}}(x)dx &= - \int u(x)D\chi_{[a,b]_{\mathbb{R}^*}}(x)dx + \left[u(x)\chi_{[a,b]_{\mathbb{R}^*}} \right]_{-\beta}^{\beta}. \end{aligned}$$

Now if $a \neq -\beta, b \neq \beta$ we have $\left[u(x)\chi_{[a,b]_{\mathbb{R}^*}} \right]_{-\beta}^{\beta} = 0$ and $D\chi_{[a,b]_{\mathbb{R}^*}}(x) = \delta_a - \delta_b$, so

$$\begin{aligned} - \int u(x) D\chi_{[a,b]_{\mathbb{R}^*}}(x) dx &= - \int u(x) (\delta_a - \delta_b) dx = \\ &= u(b) - u(a). \end{aligned}$$

If $a = -\beta, b \neq \beta$ we have $\left[u(x)\chi_{[a,b]_{\mathbb{R}^*}} \right]_{-\beta}^{\beta} = -u(-\beta)$ and $D\chi_{[a,b]_{\mathbb{R}^*}}(x) = -\delta_b$, so

$$\begin{aligned} - \int u(x) D\chi_{[a,b]_{\mathbb{R}^*}}(x) dx - u(-\beta) &= - \int u(x) (-\delta_b) dx - u(-\beta) = \\ u(b) - u(-\beta) &= u(b) - u(a). \end{aligned}$$

The case $a \neq -\beta, b = \beta$ can be proved similarly. If $a = -\beta, b = \beta$ then

$$\begin{aligned} \int Du(x)\chi_{[-\beta,\beta]}(x) dx &= \int Du(x)\tilde{1} dx = \\ - \int u(x) D\tilde{1} dx + [u(x)]_{-\beta}^{\beta} &= u(\beta) - u(-\beta), \end{aligned}$$

since $D\tilde{1} = 0$. □

Notice that $\mathbb{R} \subset \Gamma$; thus if $f \in \mathcal{C}^1(\mathbb{R})$ we have that, $\forall a, b \in \mathbb{R}$,

$$\int_a^b D\tilde{f}(x) dx = f(b) - f(a).$$

A question that arises is: does it hold, for ultrafunctions, some kind of "rule of integration by parts for continuous functions", at least for the points in Γ ? E.g., is it true that, if $u, v \in \mathfrak{U}(\mathbb{R})$ and $a, b \in \Gamma$, then

$$\int_a^b Du(x)v(x) dx = - \int_a^b u(x)Dv(x) dx + [u(x)v(x)]_a^b? \quad (14)$$

The answer is no, as a simple computation shows. Nevertheless, we have the following:

Proposition 28. *Let $u, v \in \mathfrak{U}(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})^*$, and $\gamma_n < \gamma_m \in \Gamma$. Then*

$$\int_{\gamma_n}^{\gamma_m} Du(x)v(x) dx = - \int_{\gamma_n}^{\gamma_m} u(x)Dv(x) dx + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n).$$

Proof. By (11), since $u, v \in \mathfrak{U}(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})^*$ then $Du = \tilde{u}'$ and $Dv = \tilde{v}'$. Moreover, since $\mathfrak{U}(\mathbb{R}) = \bigoplus_{j=0}^{l-1} \mathbb{I}_j$, if for every $j = 0, \dots, l-1$ we denote by P_j the orthogonal projection on \mathbb{I}_j we have

$$P_{\mathfrak{U}}u'(x) = \sum_{j=0}^{l-1} P_j(u'(x)).$$

Now, if $m = n + 1$, since u and v are continuous we have

$$\begin{aligned}
\int_{\gamma_n}^{\gamma_m} Du(x)v(x) \, dx &= \int_{\gamma_n}^{\gamma_m} P_{\mathfrak{U}}u'(x)v(x) \, dx = \\
\int_{\gamma_n}^{\gamma_m} P_n u'(x)v(x) \, dx &= \int_{\gamma_n}^{\gamma_m} u'(x)v(x) \, dx = \\
&- \int_{\gamma_n}^{\gamma_m} u(x)v'(x)dx + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) = \\
&- \int_{\gamma_n}^{\gamma_m} u(x)P_j v'(x)dx + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) = \\
&- \int_{\gamma_n}^{\gamma_m} u(x)Dv(x)dx + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n).
\end{aligned}$$

In the general case,

$$\begin{aligned}
\int_{\gamma_n}^{\gamma_m} Du(x)v(x) \, dx &= \sum_{i=n}^{m-1} \int_{\gamma_i}^{\gamma_{i+1}} Du(x)v(x) \, dx = \\
\sum_{i=n}^{m-1} \left[- \int_{\gamma_i}^{\gamma_{i+1}} u(x)Dv(x)dx + u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i) \right],
\end{aligned}$$

and since u, v are continuous we have

$$\begin{aligned}
\sum_{i=n}^{m-1} \left[- \int_{\gamma_i}^{\gamma_{i+1}} u(x)Dv(x)dx + u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i) \right] &= \\
\sum_{i=n}^{m-1} \left[- \int_{\gamma_i}^{\gamma_{i+1}} u(x)Dv(x)dx \right] + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) &= \\
- \int_{\gamma_n}^{\gamma_m} u(x)Dv(x) \, dx + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n). &\quad \square
\end{aligned}$$

The previous proposition is, in general, false if at least one between u, v is not in $\mathcal{C}^1(\mathbb{R})^*$. The reason is that, by definition, the derivative has the following expression:

$$Du(x) = P_{\mathfrak{U}}u' + \sum_{j=1}^{\ell-1} \Delta u(\gamma_j)\delta_{\gamma_j}(x),$$

and the presence of $\sum_{j=1}^{\ell-1} \Delta u(\gamma_j)\delta_{\gamma_j}(x)$ is what makes (14) to be false. Just for sake of completeness, we now show how to obtain a relaxed version of (14) by considering a different possible notion of derivative on $\mathfrak{U}(\mathbb{R})$. The relaxed version of (14) is the following: since the functions in $\mathfrak{U}(\mathbb{R})$ are piecewise \mathcal{C}^1

functions, does it hold, for ultrafunctions, an analogue of the rule of integration by parts for piecewise \mathcal{C}^1 functions? Namely, is it true that, if $u, v \in \mathfrak{U}(\mathbb{R})$ and $\gamma_n < \gamma_m \in \Gamma$, then

$$\int_{\gamma_n}^{\gamma_m} Du(x)v(x) dx = - \int_{\gamma_n}^{\gamma_m} u(x)Dv(x) dx + \sum_{i=n}^{m-1} [u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i)]? \quad (15)$$

With the operator D the answer is no. But there is a different linear operator that actually satisfies (15) :

Definition 29. We denote by $D_2u(x)$ the linear operator such that, for every $u \in \mathfrak{U}(\mathbb{R})$, we have

$$D_2u(x) = P_{\mathfrak{U}}(u'(x)).$$

Since $\mathfrak{U}(\mathbb{R}) = \bigoplus_{j=0}^{l-1} \mathbb{I}_j$, if we denote by P_j the orthogonal projection on \mathbb{I}_j , we have

$$D_2u(x) = P_{\mathfrak{U}}u'(x) = \sum_{j=0}^{l-1} P_j(u'(x)).$$

Moreover we have that, if $u(x)$ is continuous in γ_j, γ_{j+1} , then

$$Du(x) = D_2u(x)$$

on \mathbb{I}_j . In particular, if $u(x)$ is continuous in $[-\beta, \beta]$ then

$$Du(x) = D_2u(x).$$

This new linear operator is what we need to obtain the generalization to $\mathfrak{U}(\mathbb{R})$ of the rule of integration by parts for piecewise continuous functions:

Theorem 30. (Integration by parts for piecewise \mathcal{C}^1 functions) For every $u, v \in \mathfrak{U}(\mathbb{R})$ and $\gamma_n < \gamma_m \in \Gamma$ we have

$$\int_{\gamma_n}^{\gamma_m} D_2u(x)v(x) dx = - \int_{\gamma_n}^{\gamma_m} u(x)D_2v(x) dx + \sum_{i=n}^{m-1} [u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i)].$$

Proof. If $m = n + 1$ then

$$\begin{aligned} \int_{\gamma_n}^{\gamma_m} D_2u(x)v(x) dx &= \int_{\gamma_n}^{\gamma_m} u'(x)v(x) dx = \\ &- \int_{\gamma_n}^{\gamma_m} u(x)v'(x)dx + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) = \\ &- \int_{\gamma_n}^{\gamma_m} u(x)D_2v(x)dx + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n). \end{aligned}$$

In the general case we have

$$\begin{aligned} \int_{\gamma_n}^{\gamma_m} D_2 u(x) v(x) dx &= \sum_{i=n}^{m-1} \int_{\gamma_i}^{\gamma_{i+1}} D_2 u(x) v(x) dx = \\ \sum_{i=n}^{m-1} \left(- \int_{\gamma_i}^{\gamma_{i+1}} u(x) D_2 v(x) dx + u^-(\gamma_{i+1}) v^-(\gamma_{i+1}) - u^+(\gamma_i) v^+(\gamma_i) \right) &= \\ - \int_{\gamma_n}^{\gamma_m} u(x) D_2 v(x) dx + \sum_{i=n}^{m-1} [u^-(\gamma_{i+1}) v^-(\gamma_{i+1}) - u^+(\gamma_i) v^+(\gamma_i)] &= \quad \square \end{aligned}$$

In particular, since $D_2 \tilde{1} = 0$, it is immediate to prove that the following holds:

Corollary 31. (*Fundamental Theorem of Calculus for piecewise continuous functions*) For every $u \in \mathcal{U}(\mathbb{R})$ and $\gamma_n < \gamma_m \in \Gamma$ we have

$$\int_{\gamma_n}^{\gamma_m} D_2 u(x) dx = \sum_{i=n}^{m-1} [u^-(\gamma_{i+1}) - u^+(\gamma_i)].$$

Of course, the derivative D_2 has also many drawbacks, e.g. for every grid function g we have $D_2(g) = 0$. So in the following we will only consider the derivative D .

8 Ultrafunctions and distributions

In this section we briefly explain how to associate an ultrafunction to every distribution $T \in \mathcal{C}^{-\infty}(\mathbb{R})$, where

$$\mathcal{C}^{-\infty}(\mathbb{R}) = \{T \in \mathcal{D}'(\mathbb{R}) \mid \exists k \in \mathbb{N}, \exists f \in \mathcal{C}^0(\mathbb{R}) \text{ such that } T = \partial^k f\}.$$

Note that, by definition, if $T \in \mathcal{C}^{-\infty}(\mathbb{R})$ then there exists a natural number k and a function $f \in \mathcal{C}^1(\mathbb{R})$ such that:

$$T = \partial^k f. \quad (16)$$

So it is natural to introduce the following definition:

Definition 32. Given a distribution $T \in \mathcal{C}^{-\infty}(\mathbb{R})$, let k be the minimum natural number such that there exists $f \in \mathcal{C}^1(\mathbb{R})$ with $T = \partial^k f$. We denote by \tilde{T} the ultrafunction

$$\tilde{T}(x) = D^k f^*.$$

\tilde{T} will be called the ultrafunction associated with the distribution T .

Proposition 33. *For every distribution $T \in \mathcal{C}^{-\infty}(\mathbb{R})$, for every test function $\varphi \in \mathcal{D}(\mathbb{R})$ we have*

$$\int^* \tilde{T}(x) \varphi^*(x) dx = \langle T, \varphi \rangle.$$

Proof. Let us suppose that $T = \partial^k f$, where k, f are given as in Definition 32. Then, by (10), since $\varphi^*(\beta) = \varphi^*(-\beta) = 0$, we have that

$$\begin{aligned} \int^* \tilde{T}(x) \varphi^*(x) dx &= \int^* D^k f^*(x) \varphi^*(x) dx = (-1)^k \int^* f^*(x) \partial^k \varphi^*(x) dx \\ &= \left[(-1)^k \int f(x) \partial^k \varphi(x) dx \right]^* = \langle T, \varphi \rangle^* = \langle T, \varphi \rangle. \quad \square \end{aligned}$$

In the forthcoming paper [4] we will show that, actually, it is possible to define an embedding of the whole space of distributions in a particular space of ultrafunctions; this definition will be used to construct a particular algebra, related to ultrafunctions, in which the distributions can be embedded.

9 APPENDIX - Λ -theory

In this section we present the basic notions of Non Archimedean Mathematics and of Nonstandard Analysis following a method inspired by [6] (see also [1] and [2]).

9.1 Non Archimedean Fields

Here, we recall the basic definitions and facts regarding Non Archimedean fields. In the following, \mathbb{K} will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 34. *Let \mathbb{K} be an ordered field. Let $\xi \in \mathbb{K}$. We say that:*

- ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- ξ is finite if there exists $n \in \mathbb{N}$ such as $|\xi| < n$;
- ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).

Definition 35. *An ordered field \mathbb{K} is called Non-Archimedean if it contains an infinitesimal $\xi \neq 0$.*

It is easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

Definition 36. *A superreal field is an ordered field \mathbb{K} that properly extends \mathbb{R} .*

It is easy to show, due to the completeness of \mathbb{R} , that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a new notion of "closeness":

Definition 37. *We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.*

Clearly, the relation " \sim " of infinite closeness is an equivalence relation.

Theorem 38. *If \mathbb{K} is a superreal field, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the **shadow** or the **standard part** of ξ .*

Given a finite number ξ , we denote its shadow as $sh(\xi)$, and we put $sh(\xi) = +\infty$ ($sh(\xi) = -\infty$) if $\xi \in \mathbb{K}$ is a positive (negative) infinite number.

Definition 39. *Let \mathbb{K} be a superreal field, and $\xi \in \mathbb{K}$ a number. The monad of ξ is the set of all numbers that are infinitely close to it:*

$$\mathbf{mon}(\xi) = \{\zeta \in \mathbb{K} : \xi \sim \zeta\},$$

and the galaxy of ξ is the set of all numbers that are finitely close to it:

$$\mathbf{gal}(\xi) = \{\zeta \in \mathbb{K} : \xi - \zeta \text{ is finite}\}$$

By definition, it follows that the set of infinitesimal numbers is $\mathbf{mon}(0)$ and that the set of finite numbers is $\mathbf{gal}(0)$.

9.2 The Λ -limit

In this section we will introduce a superreal field \mathbb{K} and we will analyze its main properties by mean of the Λ -theory (see also [1], [2]).

We set

$$\mathfrak{X} = \mathcal{P}_{fin}(\mathfrak{F}(\mathbb{R}, \mathbb{R}));$$

we will refer to \mathfrak{X} as the "parameter space". Clearly (\mathfrak{X}, \subset) is a directed set and, as usual, a function $\varphi : \mathfrak{X} \rightarrow E$ will be called *net* (with values in E).

We present axiomatically the notion of Λ -limit:

Axioms of the Λ -limit

- **(Λ -1) Existence Axiom.** *There is a superreal field $\mathbb{K} \supset \mathbb{R}$ such that every net $\varphi : \mathfrak{X} \rightarrow \mathbb{R}$ has a unique limit $L \in \mathbb{K}$ (called the " Λ -limit" of φ .) The Λ -limit of φ will be denoted as*

$$L = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda).$$

Moreover we assume that every $\xi \in \mathbb{K}$ is the Λ -limit of some real function $\varphi : \mathfrak{X} \rightarrow \mathbb{R}$.

- **(Λ -2) Real numbers axiom.** If $\varphi(\lambda)$ is eventually constant, namely $\exists \lambda_0 \in \mathfrak{X}, r \in \mathbb{R}$ such that $\forall \lambda \supset \lambda_0, \varphi(\lambda) = r$, then

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = r.$$

- **(Λ -3) Sum and product Axiom.** For all $\varphi, \psi : \mathfrak{X} \rightarrow \mathbb{R}$:

$$\begin{aligned} \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) + \lim_{\lambda \uparrow \Lambda} \psi(\lambda) &= \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) + \psi(\lambda)); \\ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \Lambda} \psi(\lambda) &= \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) \cdot \psi(\lambda)). \end{aligned}$$

Theorem 40. The set of axioms $\{(\Lambda-1), (\Lambda-2), (\Lambda-3)\}$ is consistent.

Theorem 40 is proved in [1] and in [3].

The notion of Λ -limit can be extended to sets and functions in the following way:

Definition 41. Let $E_\lambda, \lambda \in \mathfrak{X}$, be a family of sets. We define

$$\lim_{\lambda \uparrow \Lambda} E_\lambda := \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \psi(\lambda) \in E_\lambda \right\};$$

A set which is a Λ -limit is called **internal**. In particular, if $\forall \lambda \in \mathfrak{X}, E_\lambda = E$, we set $\lim_{\lambda \uparrow \Lambda} E_\lambda = E^*$, namely

$$E^* := \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \psi(\lambda) \in E \right\}.$$

E^* is called the **natural extension** of E .

This definition, combined with axiom (Λ -1), entails that

$$\mathbb{K} = \mathbb{R}^*.$$

Definition 42. Let

$$f_\lambda : E_\lambda \rightarrow \mathbb{R}, \quad \lambda \in \mathfrak{X},$$

be a family of functions; then we define a function

$$F : \left(\lim_{\lambda \uparrow \Lambda} E_\lambda \right) \rightarrow \mathbb{R}^*$$

as follows

$$\lim_{\lambda \uparrow \Lambda} f_\lambda(\xi) := f \left(\lim_{\lambda \uparrow \Lambda} \psi(\lambda) \right);$$

where $\psi(\lambda)$ is a net of numbers such that

$$\psi(\lambda) \in E_\lambda \quad \text{and} \quad \lim_{\lambda \uparrow \Lambda} \psi(\lambda) = \xi$$

A function which is a Λ -limit is called **internal**. In particular, if $\forall \lambda \in \mathfrak{X}$,

$$f_\lambda = f, \quad f : E \rightarrow \mathbb{R},$$

we set

$$f^* = \lim_{\lambda \uparrow \Lambda} f_\lambda$$

$f^* : E^* \rightarrow \mathbb{R}^*$ is called the **natural extension** of f .

Notice that, while the Λ -limit of a constant sequence of numbers gives this number itself, the Λ -limit of a constant sequence of sets is a larger set and the Λ -limit of a constant sequence of functions is an extension of this function.

In a similar way it is possible to extend operator and functionals.

Finally, the Λ -limits satisfy the following important Theorem:

Theorem 43. (Leibnitz Principle) Let S be a set, \mathcal{R} a relation defined on S and $\varphi, \psi : \mathfrak{X} \rightarrow S$. If

$$\forall \lambda \in \mathfrak{X}, \quad \varphi(\lambda) \mathcal{R} \psi(\lambda)$$

then

$$\left(\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \right) \mathcal{R}^* \left(\lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) \right).$$

9.3 Hyperfinite sets and hyperfinite sums

Definition 44. An internal set is called **hyperfinite** if it is the Λ -limit of a net $\varphi : \mathfrak{X} \rightarrow \mathfrak{X}$.

Definition 45. Given any set $E \in \mathbb{U}$, the hyperfinite extension of E is defined as follows:

$$E^\circ := \lim_{\lambda \uparrow \Lambda} (E \cap \lambda).$$

All the internal finite sets are hyperfinite, but there are hyperfinite sets which are not finite. For example the set

$$\mathbb{R}^\circ := \lim_{\lambda \uparrow \Lambda} (\mathbb{R} \cap \lambda)$$

is not finite. The hyperfinite sets are very important since they inherit many properties of finite sets via Leibnitz principle. For example, \mathbb{R}° has the maximum and the minimum and every internal function

$$f : \mathbb{R}^\circ \rightarrow \mathbb{R}^*$$

has the maximum and the minimum as well.

Also, it is possible to add the elements of an hyperfinite set of numbers or vectors as follows: let

$$A := \lim_{\lambda \uparrow \Lambda} A_\lambda$$

be an hyperfinite set; then the hyperfinite sum is defined in the following way:

$$\sum_{a \in A} a = \lim_{\lambda \uparrow \Lambda} \sum_{a \in A_\lambda} a.$$

In particular, if $A_\lambda = \{a_1(\lambda), \dots, a_{\beta(\lambda)}(\lambda)\}$ with $\beta(\lambda) \in \mathbb{N}$, then setting

$$\beta = \lim_{\lambda \uparrow \Lambda} \beta(\lambda) \in \mathbb{N}^*$$

we use the notation

$$\sum_{j=1}^{\beta} a_j = \lim_{\lambda \uparrow \Lambda} \sum_{j=1}^{\beta(\lambda)} a_j(\lambda).$$

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